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Cofinite Induction and Artin's Theorem for Hopf Algebras

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Let H be a finite dimensional cocommutative Hopf algebra over a field K of characteristic zero. Then it is possible for H to be simple; that is, H has no proper nontrivial subHopf algebras. In particular, the Hopf algebraic analog of Artin's theorem for representations of finite groups does not hold. © 1995 Academic Press, Inc.

INTRODUCTION

Let G be a finite group and K a field of characteristic zero. Then Artin's induction theorem of group representation theory implies that up to finite multiple, the representation theory of the group G over K is determined by the representation theory of the cyclic subgroups C of G over K. More specifically, there exists an integer e, such that for every character $\chi : G \to K$, we have

$$e\chi = \sum \operatorname{Ind}_{C_i}^{G}(\chi_i),$$

where $\chi_i : C_i \to K$ is a character on the cyclic subgroup C_i of G, $\operatorname{Ind}_{C_i}^{G}(\chi_i)$ is the character on G induced from χ_i, and the sum extends over the collection of all cyclic subgroups $\{C_i\}$ of G. Since the study of cyclic, hence abelian, subgroups is generally easier than in the nonabelian case, this is a significant simplification.

The study of the representations of G over K is naturally the same as the study of KG-modules. Since KG is a finite dimensional, cocommutative Hopf algebra, it seems natural to ask whether Artin's theorem is, in fact, a Hopf algebraic result, not an exclusively group theoretic one.

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In the language we shall develop, we say that the collection of cyclic subgroup algebras $\{KC_i\}$ is cofinite in the group algebra KG. The analogous statement of Artin's theorem for a general cocommutative Hopf algebra H might then be that the collection of commutative subHopf algebras $\{J_i\}$ is cofinite for H.

We show that Artin's theorem does not generalize to all cocommutative Hopf algebras by constructing Hopf algebras which have a paucity of commutative sub-Hopf algebras. In fact, we use the cohomological classification of descent forms to construct noncommutative Hopf algebras which are simple, in that they have no proper nontrivial subHopf algebras of any kind. Since every Hopf algebra over a field is free over its subalgebras, this is trivial for prime order, by multiplicativity of rank. All the Hopf algebras constructed here are cocommutative of nonprime dimension.

In the first section we discuss some Hopf algebraic preliminaries. The second uses the group algebraic situation as a model for the Hopf algebraic analogue of Artin's theorem. The final two sections complete the construction of our simple Hopf algebras and discuss the extent to which Artin's theorem fails to generalize to all cocommutative Hopf algebras.

COFINITE INDUCTION

In this section, we discuss the basic induction theory for Hopf algebras, citing the well-known theorem of Artin for group algebras as a special case. Let H be a finite dimensional cocommutative Hopf algebra K with multiplication map $\mu : H \otimes H \to H$, unit map $\eta : K \to H$, comultiplication map $\Delta : H \to H \otimes H$, counit map $\varepsilon : H \to K$, and antipode map $\lambda : H \to H$. We use the Sweedler notation for the comultiplication,

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \qquad \text{for } h \in H.$$

Then we may define a commutative ring $\mathscr{G}(H)$ associated to H in analogy with character ring of a group over a field. For a finitely generated H-module V, let $[V]$ denote its isomorphism class and consider the abelian group $\mathscr{G}(H)$ generated by the set of all such $[V]$. Addition is defined by the equation $[V_1] + [V_2] = [V_1 \oplus V_2]$ and multiplication is defined by $[V_1][V_2] = [V_1 \otimes V_2]$, where the action of H on $V_1 \oplus V_2$ and $V_1 \otimes V_2$, respectively, is given by

$$h \cdot (v_1 \oplus v_2) = (h \cdot v_1) \oplus (h \cdot v_2),$$
$$h \cdot (v_1 \otimes v_2) = \sum_{(h)} (h_{(1)} \cdot v_1) \otimes (h_{(2)} \cdot v_2)$$

for all $h \in H$, $v_1 \in V_1$, and $v_2 \in V_2$. $\mathscr{G}(H)$ is often called the Grothendieck ring of the algebra H. Note that since H is cocommutative, it follows that $\mathscr{G}(H)$ is commutative.

For any sub-Hopf algebra J_i of H and a J_i-module W, we obtain an H-module $V = H \otimes_{J_i} W$, where H acts on $H \otimes_{J_i} W$ via the first factor. This induction process results in an additive homomorphism $\mathrm{Ind}_{J_i}^H : \mathscr{G}(J_i) \to \mathscr{G}(H)$. For a collection $\mathscr{C} = \{J_i\}$ of sub-Hopf algebras of H_j, we obtain by extension an additive homomorphism

$$\mathrm{Ind}\colon \bigoplus \mathscr{G}(J_i) \to \mathscr{G}(H),$$

where the sum extends over the given collection $\mathscr{C} = \{J_i\}$ of sub-Hopf algebras of H. This map will not always be surjective. However, let $\mathscr{G}_{\mathscr{C}}(H)$ denote the image of this map. We say that *$\mathscr{C}$ is cofinite* (*with respect to H*) if the induction map has finite cokernel. Since $\mathscr{G}(H)$ is a finitely generated abelian group, $\mathscr{C}$ is cofinite (with respect to H) if and only if the quotient group $\mathscr{G}(H)/\mathscr{G}_{\mathscr{C}}(H)$ is a torsion group, that is, if there is an integer e such that $e\mathscr{G}(H) \subseteq \mathscr{G}_{\mathscr{C}}(H)$. We call e an exponent for the collection $\mathscr{C}$. (We do not require that e be minimal.)

An important example of a cofinite collection is contained in the following theorem.

THEOREM 1.1 (Artin). *For the collection $\mathscr{C}$ of all subgroup algebras generated by the cyclic subgroups $\{C_i\}$ of G, the induction map above has finite cokernel.*

That is to say, there exists an integer e such that for every character $\chi : G \to K$, we have

$$e\chi = \sum \mathrm{Ind}_{C_i}^G(\chi_i),$$

where $\chi_i : C_i \to K$ is a character on the cyclic subgroup C_i of G, $\mathrm{Ind}_{C_i}^G(\chi_i)$ is the character on G induced from χ_i, and the sum extends over the collection $\{C_i\}$ of all cyclic subgroups of G. For a proof, see [4].

Since many of the sub-Hopf algebras with which we will deal are subgroup algebras, we discuss some results related to cofinite induction in group algebras. If $\mathscr{C}$ and $\mathscr{D}$ are two collections of subgroups of G, we say *$\mathscr{D}$ dominates $\mathscr{C}$* if every $C \in \mathscr{C}$ is contained in some $D \in \mathscr{D}$. We say that *$\mathscr{D}$ conjugately dominates $\mathscr{C}$* if every $C \in \mathscr{C}$ is conjugate to a subgroup of some $D \in \mathscr{D}$. We then have the following.

PROPOSITION 1.2. *If $\mathscr{D}$ dominates $\mathscr{C}$ and $\mathscr{C}$ is cofinite, then $\mathscr{D}$ is also cofinite.*

Proof. This is simply transitivity of induction. If C and D are subgroups of G with

$$C \subseteq D, \qquad C \in \mathscr{C}, \qquad D \in \mathscr{D},$$

then

$$\operatorname{Ind}_{KC}^{KG}(V) = \operatorname{Ind}_{KD}^{KG}(\operatorname{Ind}_{KC}^{KD}(V)),$$

so

$$\mathscr{G}_{\mathscr{C}}(KG) \subseteq \mathscr{G}_{\mathscr{D}}(KG).$$

Since the collection $\mathscr{C}$ is cofinite, there is some integer e such that $e\mathscr{G}(KG) \subseteq \mathscr{G}_{\mathscr{C}}(KG) \subseteq \mathscr{G}_{\mathscr{D}}(KG)$. Thus, $\mathscr{D}$ is also cofinite. ∎

PROPOSITION 1.3. *If C and D are conjugate subgroups of G, then*

$$\operatorname{Ind}_{KC}^{KG} : \mathscr{G}(KC) \to \mathscr{G}(KG),$$
$$\operatorname{Ind}_{KD}^{KG} : \mathscr{G}(KD) \to \mathscr{G}(KG)$$

have the same image.

Proof. In fact, we show that every KG-module induced from KC is isomorphic to a module induced from KD. Let $C = \tau D \tau^{-1}$, $\tau \in G$, and suppose V is a KC-module. Define a KD-module structure on V by letting $\delta \diamond v = (\tau\delta\tau^{-1}) \cdot v$ for $\delta \in D$, $v \in V$. We define a map

$$\Psi : KG \otimes_{KC} V \to KG \otimes_{KD} V$$
$$\sigma \otimes v \mapsto \sigma\tau \otimes v$$

for $\sigma \in \mathscr{G}$, $v \in V$.

This is clearly a KG-linear isomorphism, provided that the map is well defined. We need to show $\Psi(\sigma\gamma \otimes v) = \Psi(\sigma \otimes \gamma v)$, for $\sigma \in G$, $\gamma \in C$, $v \in V$. Computing,

$$\begin{aligned}\Psi(\sigma\gamma \otimes v) &= \sigma\gamma\tau \otimes v = \sigma\tau\delta \otimes v \qquad \text{for some } \delta \in D\\ &= \sigma\tau \otimes \delta\diamond v = \sigma\tau \otimes (\tau\delta\tau^{-1}) \cdot v = \sigma\tau \otimes \gamma \cdot v\\ &= \Psi(\sigma \otimes \gamma v), \qquad \text{as required.}\end{aligned}$$ ∎

COROLLARY 1.4. *If $\mathscr{D}$ conjugately dominates a cofinite collection $\mathscr{C}$, then $\mathscr{D}$ is also cofinite.*

Proof. Obvious, by the previous two propositions. ∎

In order to distinguish cofinite collections of sub-Hopf algebras from noncofinite ones, it is helpful to have some understanding of the structure of $\mathscr{G}(H)$. Since K has characteristic zero (actually, any characteristic prime to the rank of H over K will suffice) and H is cocommutative, it follows that the Hopf algebra H is semisimple and by Wedderburn's theorem, H decomposes into a product of simple rings $A_1 \times A_2 \times \cdots \times A_r$.

The Jordan–Hölder theorem implies that $\mathscr{G}(H)$ is freely generated as an abelian group by the finite set of isomorphism classes of the *simple* H modules, and these in turn are generated by the simple A_i-modules. Each simple ring corresponds to a minimal idempotent in H. Accordingly, we have the following.

THEOREM 1.5. *$\mathscr{G}(H)$ is a torsion-free abelian group of finite rank equal to the number of minimal idempotents in H.*

Every Hopf algebra H has a trivial representation on K via the counit map: $h \cdot k = \varepsilon(h)k$, for $h \in H$, $k \in K$. We will see later (in the proof of Theorem 3.8) that this corresponds to an idempotent in the algebra decomposition of H. Thus, we have the following observation.

PROPOSITION 1.6. *For every Hopf algebra H of (dimension greater than one) over a field K of characteristic zero,*

$$\mathrm{rank}_{\mathbf{Z}}\mathscr{G}(H) \geq 2.$$

This result will be used later to detect collections of sub-Hopf algebras that are not cofinite.

DESCENT THEORY

In this section we discuss nonabelian group cohomology and its application to classification results in descent theory. We then use this to construct Hopf algebras over a field for which the Hopf algebraic analogue of Artin's induction theorem does not hold. In fact, these Hopf algebras are simple, in the sense that they have no proper nontrivial sub-Hopf algebras of any kind, commutative or otherwise. Since a Hopf algebra defined over a field is free over all its sub-Hopf algebras [3], every Hopf algebra of prime dimension is simple, by multiplicativity of dimension. The examples presented here have nonprime rank and, therefore, are nontrivial in this respect. Most of the introductory material here may be found in [5].

Which K-spaces (algebras, Hopf algebras, vector spaces with bilinear form, etc.) W become isomorphic to V after base extension to L? That is, for which W is it true that there is an L-isomorphism of L-spaces $f : V_L \to W_L$ that respects the additional structure? We call such a W an *L-form for V.*

If L is a finite Galois extension of K with group G, we may classify the K-isomorphism classes of L-forms of V using nonabelian group cohomology. In fact,

$$\left\{\begin{matrix} K\text{-isomorphism classes} \\ \text{of } L\text{-forms of } V \end{matrix}\right\} \leftrightarrow H^1(G, A),$$

where A and the action of G on A are described as follows: Let $A = \mathrm{Aut}_L(L \otimes V)$ be the L-automorphisms of $L \otimes V$ that respect additional structure. To define an action of G on A, note first that G acts by K-algebra automorphism on $L \otimes V$ in the L factor. For $\sigma \in G$, denote this action by $op_L(\sigma)$. Then define the action of G on A by conjugation. By this we mean, given $\sigma \in G$ and $f \in A$, we define

$$\sigma \cdot f = op_L(\sigma) \circ f \circ op_L(\sigma^{-1}). \tag{2.1}$$

Now we may display the isomorphism

$$\left\{\begin{matrix} K\text{-isomorphism classes} \\ \text{of } L\text{-forms of } V \end{matrix}\right\} \to H^1(G, A).$$

Given an L-form W of V, choose an isomorphism $g : L \otimes V \to L \otimes W$. Define a 1-cocycle $p : G \to A$ by

$$p_\sigma = g^{-1} \cdot op_L \sigma \cdot g \cdot op_L(\sigma^{-1}). \tag{2.2}$$

It is straightforward to check that p is a cocycle and that a different choice of isomorphism g gives an equivalent cocycle, so the class in H^1 is well defined. That this gives a bijection between the pointed set of L-forms of V and the pointed cohomology set $H^1(G, A)$ is given in Chapter 10 of [5].

We can construct a cocommutative Hopf algebra H over K by descent from a group algebra over L using this classification. Given a finite group N, the group algebra LN is a finite dimensional L-Hopf algebra. Let $A = \text{Hopf-Aut}_L(LN)$ and let G act on A by conjugation.

CLAIM 2.3. *The action of G on A is trivial.*

Proof. A nonzero element x in a Hopf algebra is grouplike if $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$. By the definition of the Hopf algebra structure on LN, every element of N is grouplike. Now, every automorphism in A must send grouplikes to grouplikes. Since the grouplikes in a Hopf algebra are linearly independent [6, Proposition 3.2.1], the set of grouplikes in LN is precisely N, so every automorphism in A is induced by a group automorphism of N. That is,

$$A = \text{Hopf-Aut}_L(LN) \cong \text{Group-Aut}(N).$$

But now it is clear that the conjugation action of G on A is really just a composite of two cancelling actions of G on L; hence, it is trivial. ∎

Now, the fact that G acts trivially on A shows that every cocycle $p: G \to A = \mathrm{Aut}(N)$ is just a homomorphism. But such homomorphisms are precisely the actions of G on N. The equivalence condition translates in this context to the statement that two cocycles p and q are cohomologous if and only if there is a G-equivariant automorphism of N. Thus, we get the classification result for Hopf algebra forms of a group algebra:

$$\left\{\begin{matrix} K\text{-iso classes of} \\ \text{Hopf-algebra} \\ L\text{-forms of } KN \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{Equivalence classes of} \\ \text{actions of } G \text{ by} \\ \text{automorphisms on } N \end{matrix}\right\}. \tag{2.4}$$

In this context, we can identify in a simple way the Hopf algebra L-form H of KN corresponding to an action $p: G \to \mathrm{Aut}(N)$. Note that the respective actions of G on L and N gives an action of G on $LN \cong L \otimes KN$, via the tensor product representation. Set

$$H = (LN)^G = \{x \in N : \forall \sigma \in G, p_\sigma(x) = x\},$$

the fixed ring of LN under the action of G. The proof that this is actually inverse to the construction given in 2.2 is straightforward, although messy and unenlightening. To show that H is actually a form of KN, use the Morita theory, as in [1].

Remark. Clearly, H is always cocommutative (since LN is) and H is commutative if and only if N is abelian. In fact, in characteristic zero, every finite cocommutative Hopf algebra comes about this way.

PROPOSITION 2.5. *If H is a finite dimensional, cocommutative K-Hopf algebra, then H is an L-form of group algebra KN, for some Galois field extension L of K and some finite group N.*

Proof. This is essentially the statement that every finite groupscheme over a field of characteristic zero is etale. See [7, Theorems 6.4 and 11.4].

SIMPLE HOPF ALGEBRAS

In this section, we construct our simple Hopf algebras and discuss the extent to which Artin's theorem fails to generalize to a Hopf algebraic context. As a setup for the next sequence of propositions, suppose that L/K is Galois with group G, M is a subgroup of N, and we have two actions $\alpha': G \to \mathrm{Aut}(N)$, $\beta': G \to \mathrm{Aut}(M)$. Using the Galois action of G

on L, extend α' (resp. β') to an action α (resp. β) of G on $LN \cong L \otimes KN$ (resp. $LM \cong L \otimes KM$), via the tensor product representation. Let $H = (LN)^\alpha$ (resp. $J = (LM)^\beta$) be the fixed ring of the action.

PROPOSITION 3.1. *If J is contained in H, then M is α'-invariant.*

Proof. Since J is contained in H, J is α-invariant. Clearly, L is α-invariant; hence so is $L \otimes J = LM$. But, for any $\sigma \in G$, $m \in M$, we have $\alpha'_\sigma(m) \in N \cap LM = M$. So M is α'-invariant. ∎

Since we now know that M is α'-invariant, we may let $\tilde{J} = (LM)^\alpha$. By 2.5, we know that $\tilde{J}$ is an L-form of KM; that is, $L \otimes \tilde{J} \cong LM$.

COROLLARY 3.2. $J = \tilde{J}$.

Proof. $J = (LM)^\beta \subseteq LM$ and $J \subseteq H = (LN)^\alpha$. Therefore, $J \subseteq LM \cap (LN)^\alpha = (LM)^\alpha = \tilde{J}$. But

$$\dim_K J = \dim_L L \otimes J = \dim_L LM = \dim_L L \otimes \tilde{J} = \dim_K \tilde{J}.$$

Therefore, $J = \tilde{J}$. ∎

COROLLARY 3.3. *α' and β' are cohomologous actions on M.*

COROLLARY 3.4. *If α' is the trivial action of G on N, then β' is the trivial action of G on M.*

COROLLARY 3.5. *The only sub-Hopf algebras of a group algebra are the subgroup algebras.* (*Recall that* char $K = 0$.)

Proof. Suppose that J is a sub-Hopf algebra of the group algebra KN. Let L be a Galois extension of K with group G that splits J, so $LM \cong L \otimes J \subseteq L \otimes KN \cong LN$; therefore M is a subgroup of N. We contend that $J = KM$. As before, $J = (LM)^\beta$ for some action $\beta' : G \to \mathrm{Aut}(M)$. But $KN \cong (LN)^\alpha$, where $\alpha : G \to \mathrm{Aut}(N)$ is the trivial action. Thus, $\tilde{J} = (LM)^\alpha = KM$ is an L-form of J. But $J = \tilde{J}$, by 3.2. Therefore, $J = KM$, as claimed. ∎

THEOREM 3.6. *Suppose L/K is Galois with Galois group G and let $H = (LN)^\alpha$ for some finite group N and an action $\alpha' : G \to \mathrm{Aut}(N)$ of G on N. Then the sub-Hopf algebras J of H correspond to subgroups M of N that are invariant under the action of G.*

Proof. Any subgroup M of N that is invariant under the action of G gives a sub-Hopf algebra $J = (LM)^\alpha$ of $(LN)^\alpha = H$. Conversely, any sub-Hopf algebra J of H yields (on base extension to L) a sub-Hopf algebra of LN, which by the corollary above must be a subgroup algebra LM. By Proposition 3.1, M must be α-invariant. The only remaining issue is whether these are inverse bijections. But this is just the observation that $L \otimes (LM)^{\alpha} \cong LM$ (which follows from the Morita theory discussion earlier) and $(L \otimes J)^{\alpha} = (LM)^{\alpha} = J$ (which follows from the Corollary 3.2). ∎

We are now prepared to construct our simple Hopf algebras. Let L/K be a finite Galois extension of fields of characteristic zero with Galois group G. Suppose that G is a simple group (that is, one with no proper nontrivial normal subgroups) and let G act on itself by conjugation, considering $N = G$. Then we may consider the group algebra LN and the K-Hopf algebra $H = (LN)^G$. Since N is simple, N has no proper, nontrivial subgroups invariant under conjugation; hence H has no proper, nontrivial sub-Hopf algebras, by Theorem 3.6. We state this as a theorem.

THEOREM 3.7. *There exist Hopf algebras (of nonprime rank) containing no proper nontrivial sub-Hopf algebras.*

We now come to the main result of this section.

THEOREM 3.8. *A cocommutative Hopf algebra H need not possess a cofinite collection of commutative sub-Hopf algebras.*

Remark. This is in sharp contrast to Artin's theorem 1.1, which asserts that when H is a group algebra KG, the collection of cyclic subgroup algebras forms a cofinite collection.

Proof. By Theorem 1.5, the Grothendieck group $\mathscr{G}(H)$ of H is a free abelian group with rank equal to the number of minimal idempotents in H. For a Hopf algebra H over K that is split by a field extension L, the Grothendieck group of H, $\mathscr{G}(H)$, must have rank $\mathscr{G}(H) \geq 2$. This is because the group algebra $LN \cong L \otimes H$ decomposes as L-algebras into at least two factors, one coming from the idempotent

$$\frac{1}{|N|} \sum_{\nu \in N} \nu$$

of LN, corresponding to the trivial dimension one representation of N. This idempotent is fixed under any action of $G = \mathrm{Gal}(L/K)$ on N; hence this idempotent is in $(LN)^G = H$. This shows that H decomposes into at least two components as K-algebras; hence rank $\mathscr{G}(H) \geq 2$. But the collection $\mathscr{C}$ of commutative sub-Hopf algebras consists of only the trivial Hopf algebra K with all structure maps equal to the identity. Thus the image in $\mathscr{G}(H)$ of the induction map has rank one. Therefore $\mathscr{C}$ is not cofinite (with respect to H). ∎

Even when a Hopf algebra contains proper nontrivial commutative sub-Hopf algebras, it is possible that they are too sparse to provide for cofinite induction. As a general setup for our next example, suppose G is a nonabelian group and G has a unique subgroup N that is maximal among all abelian normal subgroups. As before, assume that L/K is a Galois extension with group G. Let $H = (LG)^G$, $J = (LN)^G$, with G acting on itself by conjugation. Now the collection $\mathscr{C}$ of commutative sub-Hopf algebras is $\{(LM)^G : M \text{ abelian normal in } G\}$. But N is maximal among all such so all the M's are contained in N, therefore each $(LM)^G$ is contained in J. By transitivity of induction, the image of the map

$$\text{Ind: } \bigoplus_M \mathscr{G}\big((LM)^G\big) \to \mathscr{G}(H)$$

is the same as the image of the map

$$\text{Ind: } \mathscr{G}(J) \to \mathscr{G}(H).$$

Thus to demonstrate that the collection of commutative sub-Hopf algebras is not cofinite, it suffices to show that rank $\mathscr{G}(J) <$ rank $\mathscr{G}(H)$. Now, rank $\mathscr{G}(J) \leq$ rank $\mathscr{G}(LN)$ since every idempotent in J is an idempotent in LN. Now, let $e_1, e_2, \dots, e_r$ be the minimal idempotents in LG. Each e_i need not be invariant under the G-action but since the action of G on LG is by algebra automorphisms, G permutes the e_i's. Let $X_1, X_2, \dots, X_s$ be the orbits under the action. A minimal idempotent in H is simply a sum of the elements in one of the orbits. Therefore, rank $\mathscr{G}(H) = s$. If we can show that rank $\mathscr{G}(LN) < s$, then we have

$$\text{rank } \mathscr{G}(J) \leq \text{rank } \mathscr{G}(LN) < s = \text{rank } \mathscr{G}(H),$$

so that $\mathscr{C}$ is not cofinite.

EXAMPLE 3.9. Let $G = \langle \rho, \tau : \rho^3 = 1 = \tau^2, \tau\rho\tau = \rho^2 \rangle$ be the symmetric group on three symbols and let L/K be a Galois extension with group G. Furthermore, suppose that L does not contain ζ, a primitive cube root of 1. Then the group algebra LG decomposes into a product

$$LG = Le_1 \oplus Le_2 \oplus LGe_3$$

of three simple rings, where $e_1 = \frac{1}{6}\Sigma\,\sigma$, $e_2 = \frac{1}{6}\Sigma(\text{sgn }\sigma)\sigma$, and $e_3 = 1 - e_2 - e_3$ are primitive idempotents summing to 1. It is easy to see that all of these idempotents are left invariant under the action of G so that $s = \text{rank } \mathscr{G}(H) = 3$. Let $A_3 = N$ denote the subgroup of order 3 generated by ρ. Then LN decomposes into two factors

$$Lf_1 \oplus LNf_2,$$

where f_1 is the idempotent corresponding to the trivial representation of N in L and $LNf_2 \cong L(\zeta_3)$. Thus,

$$\operatorname{rank} \mathscr{G}(J) = 2 < 3 \operatorname{rank} \mathscr{G}(H).$$

Therefore, the collection of commutative sub-Hopf algebras of H is not cofinite. ▮

Note that this example shows that Hopf algebra forms of even a solvable group algebra need not possess a cofinite family of commutative sub-Hopf algebras.

The basic difference between Hopf algebras, in general, and group algebras, in particular, seems to be that while the subalgebra generated by an arbitrary element x is obviously commutative, it is not the case that the sub-Hopf algebra generated by x need be commutative since a sub-Hopf algebra must be closed under the comultiplication as well as the multiplication. Consequently, it is possible that the elements needed to ensure such a closure would not commute with x. In the case of a group algebra, the condition that the sub-Hopf algebra be closed under comultiplication is vacuous so such a Hopf algebra is at least as rich in commutative sub-Hopf algebras as the group is rich in cyclic subgroups.

We finish with a word on classifying simple Hopf algebras (= Hopf algebras that contain no proper nontrivial sub-Hopf algebras). A subgroup M of a group N is called a characteristic subgroup if it is invariant under every automorphism of N. Since conjugation is an automorphism, a characteristic subgroup must be normal. Therefore, a simple group has no characteristic subgroups. It is easy to see that if N is a product of isomorphic simple groups, then N still has no characteristic subgroups. In fact, these are the only groups with no characteristic subgroups [2, Section 68]. We then have the following.

THEOREM 3.10. *A simple Hopf algebra H (over K) must be a form of a group algebra KN, where N is a product (possibly with only one factor) of isomorphic simple groups.*

Proof. We already know (Proposition 2.6) that there is a group N and a Galois extension L/K with group G such that $H \cong (LN)^G$ for some action $G \to \operatorname{Aut}(N)$. Furthermore, $L \otimes H \cong LN \cong L \otimes KN$. The remarks above show that it suffices to prove that N has no characteristic subgroups. But if M were such a subgroup, then the action of G on N by automorphisms would leave M invariant. Then $J = (LM)^G$ would be a proper nontrivial sub-Hopf algebra of H, contrary to hypothesis. ▮

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